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THE MATHEMATICS STUDENT

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VAGUE HISTORICAL VIEWS RELATING TO THE NEGATIVE NUMBERS.

BY G. A. MILLER,
University of Illinois.

In the history of negative numbers it is desirable to distinguish sharply between the following three points: their correct use, their correct theory and their introduction as a permanent element of mathematics. It is well-known that the Hindus used these numbers correctly and that this is the earliest such use thereof that has thus far been definitely established. On the other hand, the eighteenth century European mathematicians did not yet generally possess a correct theory of these numbers so that J. Tropfke could truly say in the third edition of his noted *Geschichte der Elementar-Mathematik*, Volume 2 (1933), page 101, that the eighteenth century suffered because a generally satisfactory introduction of these numbers was then lacking. The symbols + and — were employed in calculations and were then used at a bound to represent positive and negative numbers.

The fact that even in our day there are eminent European scientists who do not fully comprehend the theory of negative numbers is established by the following quotation from the preface of a well-known recent book: "Personally I more and more come to regard the purely formal and mathematical presentation of physical theories as a disguise and evasion of the real problems rather than as a solution of them. I have tried in other fields to show the incredible confusions, of which the whole world is now one seething example, that have followed from the invention by the Hindu mathematicians of negative quantities, and their justification from their analogy to debt, so that naturally I am not among those

who can bow down and worship the square root of minus one." F. SODDY: *The Interpretation of the Atom*, 1932.

It should be emphasized in this connection that the use of negative numbers is not fully justified from their analogy to debt. The product of two debts is not a credit. When debts are multiplied, they are multiplied by positive numbers and not by negative numbers so as to become a credit. The history of negative numbers becomes clear only by noting that the analogy of negative numbers to debt relates only to some of the properties of these numbers while it is foreign to other properties thereof. Even such an eminent mathematician as H. Cardan tried to prove in the latter half of the sixteenth century that it is incorrect to say that $(-a) \cdot (-b) = +ab$, and in the noted algebra of C. Clavius (1608) the author ascribes to the weakness of the human mind the impossibility of understanding this rule, but he did not doubt its correctness in view of the fact that it had been verified by many examples.

The theory of negative numbers with respect to the operations of addition and subtraction is much more simple than their theory with respect to the operations of multiplication and division. In the former theory, the analogy of these numbers to debt when the corresponding positive numbers represent credit and to distances in the opposite direction from those represented by the corresponding positive numbers are illuminating. In trigonometry and in analytic geometry, negative numbers are especially useful with respect to the operations of addition and subtraction. Hence it is perhaps natural that some mathematical historians have unduly emphasized this analogy in presenting the extension of the number concept so as to include negative numbers. This extension, however, implied also a correct theory of these numbers with respect to the operations of multiplication and division, and the latter theory does not seem to have been satisfactorily presented before about the beginning of the nineteenth century and it seems to be entirely due to European mathematicians who, therefore, should have the credit of having completed the introduction of the negative numbers.

Among the decidedly vague historical statements relating to the negative numbers is the following: "It was due to the influence of men like Vieta, Harriot, Fermat, Descartes and Hudde, however, that the negative number came to be fully recognized and understood." D. E. SMITH: *History of Mathematics*, Volume 2 (1925), page 259. The

190035

undue credit given to R. Descartes (1596—1650) here and in many places as regards the development of the concept of negative numbers may be largely due to the fact that in such an eminently useful work of reference as the *Encyklopadie der Mathematischen Wissenschaften*, note 18 of the first article (1898), it is stated that the actual calculation with negative numbers begins with R. Descartes and that he attributed to the same letter, sometimes a positive and sometimes a negative value. Both of these statements are inaccurate ; but on account of the place in which they appear they will doubtless continue to exert a bad influence for many years unless their inaccuracy is given wide publicity in historical articles.

In the corresponding note (149) of the French edition of this encyclopaedia, which began to appear in 1904, it is stated, on the contrary, that the systematic calculations with negative numbers are posterior to R. Descartes, and on page 100 of J. Tropfke's *Geschichte der Elementar Mathematik*, Volume 2 (1933), it is also stated that R. Descartes did not attribute sometimes a positive and sometimes a negative number to the same letter unless he had placed a dot before this letter. In fact, no definite credit is due to any of the four men, Vieta, Harriot, Fermat or Descartes for the development of the concept of negative numbers.

One of the most striking evidences of the fact that the conception of negative numbers was not fully completed before about the close of the eighteenth century, is the view that every negative number is greater than infinity which was apparently endorsed even in the second half of the eighteenth century by L. Euler, who was one of the greatest mathematicians of all times and corresponded with leading mathematicians of his day. In fact, in the latter half of the eighteenth century, there was so much written in opposition to the use of negative numbers by various European mathematicians that one might say that there was then a real crusade in Europe against the use of these numbers. It is interesting to note in this connection that the real reason which inspired the work of the noted French mathematician, L. N. M. Carnot (1753—1823), along the line of projective geometry was his aversion for negative numbers which he rejected because he thought that their use led to erroneous conclusions. This was done even in the early part of the nineteenth century. *Cf. Encyclopedie des Sciences Mathematiques*, Tome 3, Volume 2, page 3.

While the Hindus seem to have made the first correct use of negative numbers—and these numbers were used long before their theory was fully understood—the beginning of their history seems to antedate this correct use of them and to extend back at least as far as the operations with binomials such as $(a - b)^2$, where a exceeds b, by the ancient Greeks. The correctness of the fundamental rules that the product of an added number and a subtracted number is a subtracted number, and that the product of a subtracted number and a subtracted number is an added number, was established geometrically by the ancient Greeks for the cases when the minuends are greater than the subtrahends, as in the case of $(10 - 4)(8 - 3)$, and these rules were later seen to lead to correct results in many special cases where the minuends are less than the subtrahends and no case was observed where they lead to incorrect results. Hence they were gradually assumed by many to be universally true long before a proof of this fact appeared in the literature.

Hence F. Klein could almost correctly remark that "one must not think that the negative numbers are the invention of some clever man who manufactured them together with their consistency, perhaps out of the geometric representation. Rather, during a long period of development, the use of negative numbers forced itself, so to speak, upon mathematicians. Only in the nineteenth century, after men had been operating with them for centuries, was the consideration of their consistency taken up." *Elementary Mathematics from an advanced standpoint*, 1932, page 25. As a matter of fact, Kant considered negative numbers in an article as early as 1763 from practically the modern point of view and tried to make them useful in philosophy. *Cf. Zeitschrift fur mathematischen und naturwissenschaftlichen Unterricht*, 1924, page 112.

On page 23 of the volume noted in the preceding paragraph, F. Klein makes the following somewhat vague statement: "The creation of negative numbers is motivated, as you know, by the demand *that the operation of subtraction shall be possible in all cases.*" One would naturally infer from this that the idea of making subtraction always possible was a prominent mathematical idea at the time, when negative numbers began to be introduced. As a matter of fact, this is not the case. The ideas of a mathematical group and of a mathematical

domain were among the latter ideas to become explicitly vitalizing mathematical concepts. If F. Klein had said that the demand that the operation of subtraction shall be possible in all cases was one of the later powerful motives for the use of negative numbers, while earlier powerful motives therefor were their usefulness in the theory of equations, in trigonometry and in analytic geometry, he would have given here a much clearer insight into the history of these numbers.

The creation of the negative numbers was motivated by many additional things. Among these is the exponential and logarithmic motive resulting from the comparison of geometric and arithmetic series. The desirability of using different numbers to express the numerical distances from a given starting point in opposite directions made itself felt early as may be seen from the early steps towards the development of analytic geometry. Instead of agreeing with F. Soddy that incredible confusions followed from the invention of negative numbers it seems much wiser to say that the introduction of these numbers saves us from the consideration of a large number of special cases and thus it greatly simplifies the intellectual penetration into various fields. It is true that the freedom from the consideration of special cases implies, like freedom elsewhere, responsibilities as regards caution. The fact that a problem has a numerical solution does not now impose the same conditions as it did when the number concept was less general. In fact, a mathematical solution is usually only a partial solution, but it is very suggestive. Mathematics is abstract while the world is concrete, and hence there is usually much left to be done after the mathematician has contributed his part towards the solution of a problem.

The three points to which we referred at the opening of this article can now be briefly answered as follows. The Hindus used negative numbers correctly at least as early as the times of Brahmagupta, who was born in 598; and they also possessed a correct theory of these numbers as regards addition and subtraction, but there is no definite evidence that a correct theory as regards the multiplication and division of these numbers was developed before about the beginning of the nineteenth century. The latter theory is a special case of the corresponding theory relating to complex numbers and it was developed simultaneously with this more general theory. The introduction of negative numbers as a permanent element of mathematics was

inaugurated by the use of these numbers by the Hindus and was later inspired by the many simplifications which were effected by this use in various fields of mathematical developments. This introduction met with considerable opposition from time to time until a satisfactory theory of the negative numbers was developed around the beginning of the nineteenth century and thus completed their introduction more than a thousand years after this introduction was inaugurated.

The introduction of the negative numbers belongs to the extensive field of mathematical developments which are due to a failure to state all the restrictions involved in certain early proofs. The history of these developments has never been written and involves many difficulties; but the general history of mathematics is much enriched by noting that many of the early proofs apply to wider fields than the authors of these proofs had in mind. When Diophantus first stated that the product of a subtracted number by a subtracted number is an added number, he naturally had no idea of the great generality of this rule, just as J. L. Lagrange had no idea of the generality of the so-called Lagrange theorem in group theory. Many mathematical reputations are largely due to the later growth of theorems and theories rather than to their observed generality at the time when they were first announced by those to whom they are now credited. Negative numbers entered mathematics as useful servants and later were accepted as staunch citizens after their real merits became known.

ON POWERS OF NUMBERS.

By K. Subba Rao, *Vizagapatam.*

Introduction. If there exists a non-trivial solution of

$$\sum_{s=1}^{m} x_s^k = \sum_{t=1}^{n} y_t^k \quad \dots \quad (1)$$

where the x's and y's are positive integers, I write $(m) \overset{k}{=} (n)$.

If the equation (1) has an infinity of non-trivial posititive integral solutions, I write* $(m) \overset{k}{=} (n)$ *i.o.*

* *i.o.* ≡ infinitely often.

A non-trivial solution of (1) means one in which in particular

$$(x_1, \dots x_m, y_1, \dots y^n) = 1.$$

Further, I denote by $\beta = \beta(k)$, the least number β such that

$$\sum_{s=1}^{\alpha} x_s^k = \sum_{t=1}^{\beta} y_t^k \qquad (\alpha < \beta) \qquad \ldots \quad (2)$$

has at least one non-trivial solution $(x_s, y_t > 0)$. Also

$$x_1, x_2, \ldots \ldots \overset{k}{=} y_1, y_2, \ldots \ldots$$

means that $\quad x_1^k + x_2^k + \ldots \ldots = y_1^k + y_2^k \ldots \ldots$

$r'_{s,k}(n)$ denotes the number of primitive representations of n as a sum of s positive kth powers.

In this paper, I prove

Theorem 1: $\beta(6) \leqslant 5$.

This is an improvement on the known result $\beta(6) \leqslant 16$ *.

Theorem 2: There exist integers n for which $r'_{5,6}(n) \geqslant 3$. †

Theorem 3:

$$(4) \overset{6}{=} (4)$$
$$(4) \overset{6}{=} (5)$$
$$(4) \overset{6}{=} (6)$$
$$(5) \overset{6}{=} (5)$$
$$(5) \overset{6}{=} (6)$$

The second of these equations obviously implies Theorem 1.

Theorem 4: $\quad (1) \overset{6}{=} (21)$, *i.e.*‡

---

* See DICKSON: *History of the Theory of Numbers*, Vol. 2, p. 683. Also S W. P. STEEN, *Proc. Lond. Math. Soc.* (Ser. 2), Vol. 29, 333-4. It seems not impossible that $\beta(k) \leqslant k - 1$, $(k > 3)$.

† I have shown (in an as yet unpublished paper) that $r'_{5,6}(n) \geqslant 4$ has infinitely many non-trivial solutions.

‡ I have been unable to replace '21' by a smaller number, though it is likely that a much smaller number will do, perhaps '7'.

We have

$$23, 16, 11, 11 \overset{6}{=} 22, 19, 13, 12, 7$$

$$19, 19, 17, 9 \overset{6}{=} 21, 16, 15, 13, 1$$

which examples prove *Theorem 1*.

Theorem 2: There exist integers n for which $r'_{5.6}(n) \geqslant 3$, namely,*

$69, 45, 44, 38, 6 \overset{6}{=} 66, 57, 46, 20, 9 \overset{6}{=} 69, 46, 45, 30, 20$

$115, 75, 50, 23, 10 \overset{6}{=} 110, 95, 22, 19, 3 \overset{6}{=} 110, 95, 23, 15, 10$

$230, 150, 100, 69, 45 \overset{6}{=} 220, 190, 66, 57, 9 \overset{6}{=} 220, 190, 69, 45, 30$

$253, 165, 115, 75, 50, \overset{6}{=} 242, 209, 95, 33, 15 \overset{6}{=} 253, 165, 110, 95, 15$

$506, 437, 66, 57, 9 \overset{6}{=} 529, 345, 230, 45, 30 \overset{6}{=} 506, 437, 69, 45, 30$

$418, 361, 69, 45, 30 \overset{6}{=} 437, 285, 190, 66, 9 \overset{6}{=} 418, 361, 66, 57, 9$

$484, 418, 69, 45, 30 \overset{6}{=} 506, 330, 220, 579 \overset{6}{=} 484, 418, 66, 57, 9$

$506, 361, 330, 220, 57 \overset{6}{=} 484, 437, 285, 190, 66 \overset{6}{=} 484, 418, 361, 66, 57$

$529, 345, 330, 230, 220 \overset{6}{=} 484, 437, 418, 69, 66 \overset{6}{=} 506, 437, 330, 220, 69$

$529, 330, 285, 230, 45 \overset{6}{=} 506, 437, 225, 150, 69 \overset{6}{=} 529, 345, 230, 225, 150$

$529, 345, 285, 230, 190 \overset{6}{=} 506, 418, 361, 69, 57 \overset{6}{=} 506, 437, 285, 190, 69$

$418, 361, 330, 57, 45 \overset{6}{=} 437, 345, 225, 190, 150 \overset{6}{=} 437, 330, 285, 190, 45$

$506, 345, 225, 220, 150 \overset{6}{=} 484, 418, 285, 66, 45, \overset{6}{=} 506, 330, 285, 220, 45$

$506, 437, 150, 100, 69 \overset{6}{=} 529, 345, 220, 190, 30 \overset{6}{=} 529, 345, 230, 150, 100$

---

* These examples have been derived from the example $22, 19, 3 \overset{6}{=} 23, 15, 10$. I should add that the existence of integers n for which $r'_{k,k}(n) \geqslant 2$ $(k > 6)$ has not been proved.

The following examples prove Theorem 3 :—

$20, 13, 13, 9 \stackrel{6}{=} 19, 17, 12, 5$

$19, 19, 17, 9 \stackrel{6}{=} 21, 16, 15, 13, 1$

$23, 16, 11, 11 \stackrel{6}{=} 22, 19, 13, 12, 7$

$19, 17, 10, 1 \stackrel{6}{=} 20, 14, 9, 7, 4, 3$

$17, 13, 13, 1 \stackrel{6}{=} 16, 16, 7, 7, 3, 3$

$19, 18, 17, 5 \stackrel{6}{=} 21, 15, 14, 8, 8, 3$

$21, 16, 14, 11, 11 \stackrel{6}{=} 22, 7, 7, 3, 2$

$20, 19, 12, 7, 6 \stackrel{6}{=} 21, 16, 15, 8, 2$

$18, 18, 13, 13, 5 \stackrel{6}{=} 20, 15, 11, 9, 2$

$18, 18, 11, 5, 5 \stackrel{6}{=} 19, 15, 15, 2, 2$

$20, 11, 11, 9, 5 \stackrel{6}{=} 19, 15, 13, 13, 2$

$20, 14, 11, 1, 1 \stackrel{6}{=} 19, 17, 10, 10, 7$

$16, 16, 10, 7, 3 \stackrel{6}{=} 17, 14, 12, 5, 4$

$21, 13, 13, 12, 11 \stackrel{6}{=} 20, 17, 15, 9, 7$

$19, 10, 10, 8, 6 \stackrel{6}{=} 18, 14, 13, 12, 2, 2$

$19, 16, 16, 6, 2 \stackrel{6}{=} 18, 18, 14, 13, 8, 4$

$19, 19, 13, 12, 11 \stackrel{6}{=} 20, 16, 15, 15, 7, 1$

$19, 16, 13, 3, 2 \stackrel{6}{=} 20, 12, 10, 9, 7, 5$

$19, 16, 16, 4, 2 \stackrel{6}{=} 20, 14, 13, 12, 10, 8$

$19, 19, 16, 6, 1 \stackrel{6}{=} 20, 18, 14, 13, 9, 5$

$16, 13, 11, 9, 2 \stackrel{6}{=} 15, 15, 10, 7, 4, 4$

$22, 7, 3, 3, 2 \stackrel{6}{=} 21, 15, 14, 13, 12, 10$

$20, 20, 13, 4, 3 \stackrel{6}{=} 22, 15, 14, 9, 2, 2$

$20, 20, 13, 9, 4 \stackrel{6}{=} 22, 15, 14, 10, 6, 5$

$20, 20, 10, 1, 1 \stackrel{6}{=} 22, 14, 13, 12, 8, 5$

$13, 13, 13, 12, 12 \stackrel{6}{=} 15, 14, 10, 9, 3, 2$

$$23, 16, 15, 13, 2 \overset{6}{=} 22, 20, 12, 9, 7, 5$$

$$18, 17, 13, 10, 5 \overset{6}{=} 19, 16, 7, 6, 4, 3$$

$$19, 17, 9, 8, 2 \overset{6}{=} 20, 13, 12, 7, 6, 1$$

$$19, 16, 13, 6, 1 \overset{6}{=} 18, 17, 14, 12, 5, 5$$

$$22, 18, 17, 13, 5 \overset{6}{=} 23, 16, 15, 7, 6, 4$$

$$36, 13, 8, 8, 5 \overset{6}{=} 32, 32, 18, 9, 7, 4$$

$$40, 36, 19, 10, 6 \overset{6}{=} 42, 30, 20, 18, 13, 4$$

We have

$$(6t^6+1)^6 = (6t^6 - 1)^6 + 2\,(6t^5)^6 + 2\,(4t^3)^6 + 7\,(2t^3)^6 + (2t)^6 + 8\,(t^6)$$

The right hand side contains *twenty-one* sixth powers and Theorem 4 is proved.

[Received 27-11-1933.]

A BIQUADRATE AS THE SUM OF FOUR BIQUADRATES.*

BY S. SASTRY,

Lecturer in Mathematics, Benares Hindu University.

1. In confirmation† of Euler's conjecture that a fourth power can be the sum of four fourth powers R. Norrie found in 1910 by a series of special assumptions the single result ‡

$$353^4 = 30^4 + 120^4 + 272^4 + 315^4.$$

I shall give here a simple method of deriving this interesting result. It may be possible to use this method to derive other examples of a fourth power as the sum of four fourth powers.

---

* I wish to express my sincere thanks to Dr. S Chowla for reading the manuscript and for suggesting corrections.

† DICKSON : *History of the theory of Numbers*, Vol. 2, p. 652. Norrie's paper has been inaccessible to me.

‡ In a paper communicated to the London Mathematical Society, I gave the corresponding example for fifth powers, namely,

$$107^5 = 43^5 + 7^5 + 57^5 + 80^5 + 100^5.$$

Recently* Chowla and I have proved that a fifth power can be the sum of five positive fifth powers infinitely often. The corresponding result has not yet been proved for fourth powers.

2 We have

$$(a+b+c)^4-(a+b)^4=c(2a+2b+c)\,\{\,2a^2+2b^2+c^2+4ab+2ac+2bc\,\}\,.$$

Changing a to a^4, b to b^4 and c to c^4, we have

$$(a^4+b^4+c^4)^4-(a^4+b^4)^4$$

$$= c^4\,(2a^4+2b^4+c^4)\,\{\,2a^8+2b^8+c^8+4a^4b^4+2a^4c^4+2b^4c^4\,\} \qquad \ldots \quad (1)$$

My method is to express $2a^4+2b^4+c^4$ as a fourth power and $2a^8+2b^8+c^8+4a^4b^4+2a^4c^4+2b^4c^4$ as the sum of three fourth powers.

The first is always possible by means of E. Fauquemberge's identity†

$$(4x^4+y^4)^4=(4x^4-y^4)^4+2\,(4x^3y)^4+2\,(2xy^3)^4$$

if we take $a=4x^3y$, $b=2xy^3$ and $c=4x^4-y^4$. The second is also possible in the particular case $x=y=1$, *i.e.*, when $a=4$, $b=2$, and $c=3$. In this case,

$$2a^8+2b^8+c^8+4a^4b^4+2a^4c^4+2b^4c^4$$

$$=2\,(a^2)^4+2\,(b^2)^4+(c^2)^4+3\,(ab)^4+2\,(ac)^4+2\,(bc)^4-b^4+b^4+(ab)^4$$

$$=(2.16^4+2.4^4+9^4+3.8^4+2.12^4+2.6^4-2^4)+2^4+8^4$$

$$=\qquad 21^4 \qquad +2^4+8^4$$

In this case, (1) becomes $353^4-272^4=3^4.5^4\,(21^4+2^4+8^4)$,

that is $$35=272^4+315^4+30^4+120^4.$$

---

* To appear elsewhere.

† See DICKSON : *History of the theory of Numbers*, Vol. 2, p. 650.

A BRIEF SKETCH OF HINDU TRIGONOMETRY.

By D. A. Somayajulu,
Research Student, University of Madras.

In ancient India, Trigonometry was developed as an adjunct to Astronomy. Hence we find that in works on Astronomy, the author before proceeding to give the Grahasphutam (the process of finding the true places of planets from the mean) tells us how to find the sines of various angles. There is this difference, however, that the "sine" of the Hindus does not represent a ratio, as with us, but a linear measure ment. Their standard unit of measuring sines of angles, or what is the same thing, of measuring lengths, is the radius of the circle ; and since the radius of any circle is equal to the length of an arc substending an angle of $57^\circ\ 17'\ 45'' = 3438$ minutes approximately (radian), the Hindus make their radius equal to 3438′, and whenever they refer to त्रिज्या in their works this length of 3438′ is meant by them.*

Now, in the circle with centre C and radius CA equal to 3438′, the half-chord AM represents the Hindu Sine of the angle $A\hat{C}M$, so that to get the modern sine, we have to divide the former by $R = 3438'$; and similarly to get the modern trigonometric formula out of those of the Hindus, we have to substitute R sin A, R cos A respectively for Sin A and Cos A. Thus the Hindu Sine of 90° is 3438. It will be seen that the word त्रिज्या in Sanskrit meaning 'Radius,' has derived its meaning from the fact that it is the ज्या or sine of three rasis (rasi = 30°, since the Ecliptic is divided into twelve equal divisions called rasis).

Now, since the Hindus knew Pythogoras theorem (Playfair and other orientalists assert that Pythogoras got this theorems and some others also from the Hindus), they were familiar with most of the elementary trigonometrical relations derivable from right-angled triangles, such as $\text{Cos A} = \sqrt{R^2 - \text{Sin}^2 A}$; $\text{Sin } 30^\circ = \text{Cos } 60^\circ = R/2$; $\text{Sin } 45 = R/\sqrt{2}$, etc., etc., (*vide* slokas 4, 6, under *Jyotpatti* Goladhyaya, *Sidhanta Sriomani*). Bhaskara also gives the Sines of 36° and 18°, in the slokas.

त्रिज्याकृतीषुघातात् त्रिज्याकृतिवर्गघातस्य, ।
मूलोनादष्टहृतान्मूलं षट्त्रिंशदंशज्या ॥ (7)

---

* In this paper, we shall write the Hindu Sine and Cosine with capital initials and the usual Sine and Cosine (which are ratios) with small letters *s* and *c*.

षट्त्रिंशदंशजीवा तत्कोटिज्याकृतेषूणाम
त्रिज्याकृतीषुघातात् मूलं त्रिज्योनितं चतुर्भक्तम
अष्टादशभागानाम् जीवा स्पष्टा भवत्येवं ॥ (9)

(*Jyotpatti*)

which means " Deduct the square-root of five times the fouth-power of the radius from five times the square of the radius, and divide the remainder by eight ; the square-root of the quotient will be the Sine of 36° or the Sine of 54°" or in symbols

$$\text{Sin } 36^\circ = \sqrt{\frac{5R^2 - \sqrt{5R^4}}{8}}$$

" To get the Sine of 18°, deduct the radius from the square-root of the product of the radius and five, and divide the remainder by 4 ; the quotient will be found to be the exact Sine of 18°, or in symbols."

$$\text{Sin } 18^\circ = (\sqrt{5R} - R) / 4$$

The more general relations between the trigonometric functions are given by Bhaskara in the following words :

क्रमोत्क्रमज्याकृतियोगमूलाद्दलं तदर्धांशकशिञ्जिनीस्यात् ।
त्रिज्योत्क्रमज्यानिहतेर्दलस्य मूलं तदर्धांशकशिञ्जिनी वा ॥ (10)
त्रिज्याभुजज्याहतिहीनयुक्ते त्रिज्याकृती तद्दलयोः पदेस्तः ।
भुजोनयुक्तत्रिभखण्डयोः ज्ये (12)

(*Jyotpatti*)

which means

$$\text{Sin } \frac{A}{2} = \frac{1}{2}\sqrt{\text{Sin}^2 A + \text{Versin}^2 A} = \sqrt{\frac{R}{2} \text{ versine } A}$$

$$\text{Sin } \left(45 + \frac{A}{2}\right) = \sqrt{R^2 + R \text{ Sin } A} \Big/ \sqrt{2} :$$

$$\text{Sin } \left(45 - \frac{A}{2}\right) = \sqrt{R^2 - R \text{ Sin } A} \Big/ \sqrt{2}$$

which correspond, in modern notation to

$$R \sin\frac{A}{2} = \frac{1}{2}\sqrt{R^2 \sin^2 A + (R - R\cos A)^2} = \frac{R}{2}\sqrt{2 - 2\cos A},$$

and

$$R \sin\left(45 + \frac{A}{2}\right) = \sqrt{R^2 + R^2 \sin A} \Big/ \sqrt{2} = R\sqrt{1 + \sin A} \Big/ \sqrt{2}.$$

Again when A and B are two arcs, Bhaskara gives the sine of their difference

$$\text{Sin}\ \tfrac{1}{2}(A - B) = \tfrac{1}{2}\left\{(\text{Sin}\ A - \text{Sin}\ B)^2 + (\text{Cos}\ A - \text{Cos}\ B)^2\right\}^{\frac{1}{2}}$$

in the following sloka :

' यद्दोर्ज्ययोरतंरमिष्टयोर्यत् कोटिज्ययोः तत् कृतियोगमूलं,
दलीकृतं स्यात् भुजयोर्वियोगखंडस्य जीवैवमनेकधावा " (13)

(*Jyotpatti*)

Bhaskara next gives the calculation of sines without having recourse to the extraction of roots in the following sloka.

दोर्ज्याकृतिर्व्यासदलार्थभक्ता लब्धत्रिमौर्व्योः
विवरेण तुल्या दोःकोटिभागांतराशिञ्जिनी स्यात्

which is equivalent to the formula $\text{Sin}\,(2A - 90) = (R^2 - 2\,\text{Sin}^2 A) / R$ ' In this way several sines may be found" he says and then proceeds. 'to give the rules for finding the sines of every degree from 1° to 90° In doing so, he utilizes the formulae

$$R \sin (A \pm B) = \text{Sin}\,A \cos B \pm \text{Cos}\,A \sin B.$$

which are given in the following verses

" चापयोरिष्टयोः दोर्ज्ये मिथः कोटिज्यकाहते,
त्रिज्याभक्ते तयोरैक्यं स्यात् चापैकस्य दोर्ज्यका,
चापान्तरस्य जीवा स्यात् तयोरन्तरसंमिता,
अन्यज्यासाधने सम्यगियं ज्याभावनोदिता."

The process of finding the sines of different angles, given above by Bhaskara, is termed by him "प्रतिभागज्यकाविधिः" which means "the

method of finding the sines of every degree," and can be summed up as follows. Suppose we are to find the Sine of 35° . We start with the angle which is nearest to the given angle whose sine is known; for instance 30°; then we use the formula given by the sloka

"स्वगोंगेषु षडंशोनवर्जिता भुजशिञ्जिनी,
कोटिज्या दशभिः क्षुण्णा त्रिसप्तेषुविभाजिता,
तदैक्यमग्रजीवा स्यात् अंतरंपूर्वशिञ्जिनी."

"Subtract 1/6569th part of Sine 30° from the same: multiply the Cosine of 30° by 10 and divide by 573; add or subtract the resulting values; you will get Sin (30 + 1°) or sin (30 — 1°)." We proceed thus, and get the sine of 35°. This formula is equivalent to

$$R \text{ Sin } 31^\circ = \text{Sin } 30^\circ \text{ Cos } 1^\circ + \text{Cos } 30^\circ \text{ Sin } 1^\circ.$$

Here the value of Sine 1°, namely 60′, is utilized and calculations are simplified taking advantage of the constant factors Sin 1° and Cos 1°.

The method given in *Surya Siddanta* for obtaining the sines of different angles is different from this. There the quadrant is divided into 24 equal parts, each containing 3°45′ and the Sine of each is calculated starting with the initial Sine, namely that of 3°45′ which is taken to be 225′. This may be accounted for in two ways. It may be that the Hindu Siddhantis perceived that the sine of a very small angle would be equal to its circular measure; or, they might have actually calculated Sin 3°45′ starting from Sin 30°. Judging from the fact that they knew the formula Sin ½ A = $\sqrt{\tfrac{1}{2} \text{R versin A}}$, it is more probable that the ancients actually calculated the Sin of 3°45′ and finding that it is very nearly equal to 225, adopted this as the initial sine. The formula adopted, is.

$$\text{Sin } (n+1) \text{ A} = \text{Sin } n\text{A} + \text{Sin A} - \left(\sum_{r=1}^{r=n} \text{Sin } r\text{A} \right) / \text{Sin A} \ldots \quad \text{(i)}$$

contained in the following slokas of *Surya Siddhanta*, Chapter 2.

"राशिलिप्ताष्टमोभागः प्रथमं ज्यार्थमुच्यते,
तत्तद्विभक्तलब्धोनमिश्रितं तत् द्वितीयकं,
आद्येनैवं क्रमात्पिंडान् त्तवालब्धोनसंयुताः
खण्डकास्युश्चतुर्विंमृश ज्यार्थपिण्डाः क्रमादमी."

The formula may be obtained thus

$$\text{Sin}\,(n+1)\,A + \text{Sin}\,(n-1)\,A = \text{Sin}\,nA\,(\text{Sin}\,2A\,/\,\text{Sin}\,A).$$

Replacing n by $n-1$, $n-2$, ... 1 in succession and adding all the results we have

$$\text{Sin}\,(n+1)\,A + 2\sum_{r=1}^{n} \text{Sin}\,rA - \text{Sin}\,nA - \text{Sin}\,A$$

$$= (\sum_{r=1}^{n} \text{Sin}\,rA)\,\text{Sin}\,2A\,/\,\text{Sin}\,A.$$

$$\text{Sin}\,(n+1)\,A = \text{Sin}\,nA + \text{Sin}\,A - (\sum_{1}^{r} \text{Sin}\,rA)(2\text{Sin}\,A - \text{Sin}^2 A)\,/\,\text{Sin}\,A.$$

Now, finding that Sin A = 225′ and Sin 2A = 449′ (A = 3°45′) which values may have been obtained from Sin 30° as remarked above, they saw that the factor 2 Sin A — Sin 2A reduced to un ty so that the formula now takes the form given above.

Indeed, we can calculate the sines of consecutive angles from 1° upto the 90° utilizing the same formula in a different way. Instead of assuming Sin 30°45′ = 225′, we will be nearer the truth by assuming Sin 1° = 60′. Hence calculating Cos 1° from the formula $\text{Cos}\,\theta = \sqrt{R^2 - \sin^2\theta}\,/\,R$, we find Cos 1° = 3437·47′.

∴ Sin 2° = (2 Sin 1° Cos 1°) / 3438 = 2 × 60 × 3437·47 / 343·8 = 119·9815′

Hence 2 Sin 1° — Sin 2° = ·0185′. Hence in the formula making A = 1°, and the factor 2 Sin 1° Sin 2° = ·0185′, the successive sines of 1°, 2°, 3°, etc., can be calculated easily. Thus

Sin 1° = 60′	Sin 6° = 359·353′	Sin 11° = 655·765′
Sin 2° = 119·9815′	Sin 7° = 418·965′	Sin 12° = 714·562′
Sin 3° = 179·926′	Sin 8° = 478·447′	Sin 13° = 773·137′
Sin 4° = 239·815′	Sin 9° = 537·781′	Sin 14° = 831·490′
Sin 5° = 299·630′	Sin 10° = 596·782′	Sin 15° = 889·70′

The approximation will be found correct to five decimal places.

Spherical Trigonometry.

In spherical trigonometry, the Hindu Siddhantis were conversant with Napier's rules concerning right-angled spherical triangles and with various other formulae such as

$$\cos c = \cos a \cos b + \sin a \sin b \cos C.$$
$$\sin c / \sin C = \sin b / \sin B = \sin a / \sin A, \text{ etc.}$$

Their methods may be best illustrated by a few examples, which have a bearing on Astronomy.

EXAMPLE 1. To find the Sun's altitude when the azimuth is 45°, given only one observation, namely, that of the rising amplitude* and *palabha*.† The rule given in the *Surya Siddhanta* reads :

त्रिज्यावर्गार्धतोग्रज्यावर्गोनाद् द्वादशाहतात् ।
पुनर्द्वादशनिघ्नाच्च लभ्यते यत्फलं बुधैः ॥
शङ्कुवर्गार्धसंयुक्तविषुवद्वर्गभाजितात् ।
तदेवकरणीनाम तांपृथक् स्थापयेत् बुधः ॥
अर्कघ्नी विषुवच्छायाग्रज्यया गुणिता तथा ।
भक्ता फलाख्यं तद्वर्गसंयुक्त करणीपदं ॥
फलेन हीनसंयुक्तं दक्षिणोत्तरगोलयोः ।
तत्रिज्यावर्गविश्लेषान्मूलं दृग्ज्याभिधीयते ॥

Surya Siddhanta, Chapter 3, *Slokas*, 28, 29, 30, 32.

" Sin (Sun's altitude) $= (\text{Karani} + \text{phala}^2)^{\frac{1}{2}} \mp \text{phala}$,

where

$$\text{Karani} = 12^2 \left(\tfrac{1}{2} R^2 - \text{Sin}^2 A\right) / \left(\tfrac{1}{2} 12^2 + \text{palabha}^2\right), \text{ and}$$
$$\text{phala} = (12 \text{ palabha} \times \text{Sin } A) / \left(\tfrac{1}{2} 12^2 + \text{palabha}^2.\right)$$

where (A = 90 —azimuth)."

To prove the truth of this formula, let us take the spherical triangle PZS, in which P is the pole, Z the zenith and S the sun. If

$PS = 90 - \delta$; $PZ = 90 - \phi$: $ZS = z$; $\angle PZS = 360° - \alpha$;

we have $\sin \delta = \cos z \sin \phi + \sin z \cos \phi \cos \alpha$

* Amplitude is used by the Hindus in place of 90°—azimuth.

† *Palabha* is the shadow cast by a gnomon of 12 units height at noon on a day of an equinox.

3

where α is measured from the north point, so that the amplitude is here 180 — α. Transforming this formula into the Hindu one, it becomes

$$R^2 \operatorname{Sin} \delta = R \operatorname{Cos} z \operatorname{Sin} \phi + \operatorname{Sin} z \operatorname{Cos} \phi \operatorname{Cos} \alpha.$$

Substituting in this

$$\sin \phi = (\text{palabha} \times R) \,/\, \text{hypotenuse}^* = pR/H$$
$$\operatorname{Cos} \phi = (R \times \text{gnomon}) \,/\, \text{hypotenuse} = Rg/H$$

and $\alpha = 180 - 45^\circ = 135^\circ$,

we have $$\operatorname{Sin} \delta = \frac{p \operatorname{Cos} z}{H} - \frac{g \operatorname{Sin} z}{H \sqrt{2}}.$$

But, from the spherical triangle SnP (see figure next page) where S is the rising Sun, n the north point of the horizon, Sn = 90 — A where A is his rising amplitude, and SP = 90 — δ, Pn = ϕ,

$$\operatorname{Sin} \delta = \operatorname{Cos} \phi \operatorname{Sin} A/R = g \operatorname{Sin} A/H.$$

Substituting this value of Sin δ in the above formula

$$\frac{g \operatorname{Sin} A}{H} = \frac{p}{H} \operatorname{Cos} z - \frac{g \operatorname{Sin} z}{H \sqrt{2}}.$$

But since $\operatorname{Sin}^2 z = R^2 - \operatorname{Cos}^2 z$, this reduces to

$$p \operatorname{Cos} z - g \sqrt{R^2 - \operatorname{Cos}^2 z} \,/\, \sqrt{2} = g^2 \operatorname{Sin} A$$

i.e., $$(p^2 + \tfrac{1}{2} g^2) \operatorname{Cos}^2 z - 2 \operatorname{Cos} z \,(pg \operatorname{Sin} A) = g^2 (\tfrac{1}{2}R^2 - \operatorname{Sin}^2 A)$$

i.e., $$\operatorname{Cos}^2 z - 2 \operatorname{Cos} z \times \text{phala} = \text{Karani}$$

by substituting phala and Karani respectively for

$$\frac{12\, p \operatorname{Sin} A}{72 + p^2} \text{ and } \frac{g^2 (\tfrac{1}{2}R^2 - \operatorname{Sin}^2 A)}{72 + p^2}; \qquad \text{since } g = 12$$

i.e., $\operatorname{Cos} z = \sqrt{\text{Karani} + \text{Phala}^2} \mp \text{phala}$ as is given in the *Surya Siddhanta*.

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* The fundamental triangle used by the Hindus is the right-angled triangle formed of the gnomon, its shadow and the hypotenuse. It is called the प्रधानाक्षक्षेत्र Thus, if AB represents the gnomon, BC the equinoctial shadow, and AC the hypotenuse, $\angle A = \phi$ represents the latitude ; hence $\sin \phi/R$ = palabha / hypot.

EXAMPLE 2. To find the Sun's altitude when h (hour angle), δ and ϕ are given. The rule given in the *Surya Siddhanta* is

"त्रिज्योदक् चरजायुक्ता याम्यायां तद्विवर्जिता, अंत्या नतोत्क्रमज्योना स्वाहोरात्रार्धसंगुणा. त्रिज्याभक्तो भवेत् छेदो लम्बज्याघ्नोऽथभाजितः त्रिभज्यया भवेत् शंकुःतद्वर्गं परिशोधयेत् त्रिज्यावर्गात् पदं दृग्ज्या, ...

i.e.,

$$R \pm \text{Sin (chara)} = \text{Antya};$$

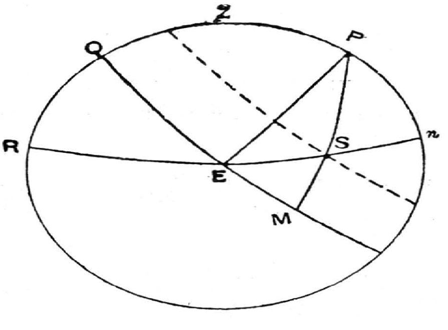
$$(\text{Antya} - \text{Vers } h) \text{ Cos } \delta / R = \text{cheda};$$

$$(\text{cheda} \times \text{Cos } \phi) / R = \text{Sanku} = \text{Sin (Sun's altitude)}.$$

Hence

$$\text{Sin (Sun's altitude)} = \{ R \pm \text{Sin (chara)} - \text{Vers } h \} \text{ Cos } d \text{ Cos } \phi / R^2$$

The arc EM in the figure, which is the portion of the equator intercepted between the declination circle of the Sun while rising and the fixed circle PE, is called *chara* or Ascential difference. The fixed



circle PE is called उन्मण्डल or the rising circle. From the right-angled spherical triangle EMS in which MS is the declination of the Sun, EM the Ascential difference and the angle SEM = 90 — ϕ, we have by Napier's rules sin (chara) = $\pm$ tan δ tan ϕ, or with the Hindu

notation

$$\text{Sin (chara)} = \pm R \tan \delta \tan \phi. \quad (1)$$

Also from the triangle PZS

$$\cos z = \sin \delta \sin \phi + \cos \delta \cos \phi \cos h$$

which in Hindu Trigonometry becomes

$$\text{Cos } h = R^2 \text{ Cos } z / \cos \phi \text{ Cos } \delta \mp R \tan \phi \tan \delta. \quad (2)$$

Hence by addition from equations (1) and (2)

$$\text{Cos } h \pm \text{Sin (chara)} = R^2 \text{ Cos } z / \text{Cos } \phi \text{ Cos } \delta.$$

$$\therefore \quad \{ \text{Cos } h \pm \text{Sin (chara)} \} \text{ Cos } \phi \text{ Cos } \delta / R^2 = \text{Cos } z$$

which is also Sin (Sun's altitude). This is the formula given in the *Surya Siddhanta*.

EXAMPLE 3. To find the true *Valana*, (*i.e.*, the angle which determines the points of the disc of the eclipsed body where the eclipse commences and where it ends).

In *Surya Siddhanta* the first Sloka in Chapter 9, remarks

नच्छेद्यकमृते यस्मात् भेदा ग्रहणयोः स्फुटाः ।
ज्ञायंते तत् प्रवक्ष्यामि छेद्यकज्ञानमुत्तमं ॥

i.e., "the phases of the Eclipses cannot be understood without knowing the method of projection (छेद्यकविधि)."

The angle called *Sphuta Valana* is the angle between Mn and MK where M is the centre of the eclipsed body, the Sun or Moon, P the celestial pole, and K the Kadamba or the pole of the Ecliptic and n the north point of the horizon. This angle is equal to the sum or difference of the angles PMn and KMP. The $\angle$PMn is called the *Aksha Valana* and the $\angle$KMP is called the *Ayana Valana*.

From the triangle PMn,

$$\sin(\text{Aksha Valana}) / \sin \phi = \sin PnM / \cos \delta$$

and from the triangle KMP,

$$\sin(\text{Ayana Valana}) / \sin \omega = \cos \lambda / \cos \delta$$

(where λ is the longitude of the centre of the eclipsed body M; and ω = obliquity of the ecliptic = $24°$ according to the Hindu works).

The *Surya Siddhanta* gives the same formulae, except that it has the radius of the diurnal circle in the place of Cos δ. But both are identical because, if E′Q′ represent an arc of the diurnal circle parallel to the Equator, $E'Q' / EQ = \text{Cos}\,\delta / R$.

But $$\frac{E'Q'}{EQ} = \frac{\text{diurnal circle}}{\text{Equator}} = \frac{2\pi \times \text{radius of the diurnal circle}}{2\pi R}.$$

Hence Cos δ is equal to the radius of the diurnal circle which is called द्युज्या in Sanskrit. The sum of the angles Aksha Valana and Ayana Valana determines the angle which gives the means of projecting the line of the Ecliptic upon the disc of the Eclipsed body, the Sun or the Moon.

GLEANINGS

19. The boy or girl should be accustomed to use language like a rapier, should despise a clumsiness in inference as if it were a foul in a game. The classes should collect blunders, disingenuous statements and false conclusions from public discussions. They should botanise for errors and bring the precious finds to the class room for dissection. They will also go as far as their willingness and aptitudes will take them in the exact and rigid reasoning processes of Mathematics

H. G. WELLS: *The Work, Wealth and Happiness of Mankind*, page 759.

20. What is physical is subject to the laws of mathematics and what is spiritual to the laws of God, and the laws of Mathematics are but the expression of the thoughts of God.

THOMAS MILL: The uses of Mathesis; Bibliotheca Sacra,
Extracted from *Memorabilia Mathematica*, No 275.

21. A story is told by Einstein against himself that comes in amusingly here. He is a great mathematician but he is not a ready reckoner, and during the crisis of the mark he thought that a tram conductor had given him back too much change—a hundred thousand marks or so—and had to be convinced of his error. "Everybody," said the kindly tram conductor, "hasn't the gift of calculation with these big figures. I must not take advantage of you ..." And Laplace, one of the greatest mathematicians the world has ever known, was dismissed from the Ministry of the Interior by Napoleon for the grossest incompetence. But our organising world requires these exceptional individuals far more than it does an endless multitude of fairly good-all-round people

H. G. WELLS: *The Work, Wealth and Happiness of Mankind*, page 680

NOTES AND DISCUSSIONS.

The Editor welcomes for publication under this heading, brief discussions of interesting problems, critical comments, and suggestions likely to be helpful in the class-room.

Congruence Properties of Partitions.

Let $p(n)$ denote the number of unrestricted partitions of n. Ramanujan* conjectured that

If $\delta = 5^a 7^b 11^c$ *and* $24\lambda \equiv 1 \pmod{\delta}$, *then*

$$p(\lambda),\ p(\lambda + \delta),\ \ldots\ \ldots\ \equiv 0 \pmod{\delta}.$$

A recent table † by H. Gupta gives

$$p(243) = 133\ 978\ 259\ 344\ 888.$$

Hence

Ramanujan's conjecture is false for $\lambda = 243,\ \delta = 343.$

S. CHOWLA.

[Received 24-2-34.]

On a certain Arithmetical Function.

Let $n = p_1^{\alpha_1} \ldots p_r^{\alpha_r}$ where the p's are distinct primes,

$$\lambda(1) = 1, \qquad \lambda(n) = (-1)^{\alpha_1 + \cdots + \alpha_r} \quad (n \geqslant 2),$$

$$L(x) = \sum_{n \leqslant x} \lambda(n).$$

Pólya‡ has verified that for $2 \leqslant x \leqslant 1500$,

$$L(x) \leqslant 0. \qquad \ldots \quad (1)$$

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* *Proc. Camb. Phil. Soc.*, XIX (1919), 207—210.

† This table gives $p(n)$ for $n \leqslant 300$. With the help of Macmohan's table (extending to $n \leqslant 200$) I have verified Gupta's table upto $n = 243$.

‡ *Jahresberichte der Deutsche Math. Vereinigung*, 28, 1919, 38—40.

I have tabulated* the values of $L(x)$ for all $x \leqslant 10000$ *and find that* (1) *is true up to this limit.*

The values of $L(x)$ for $x = 1000, 2000, \ldots, 10000$ are $-14, -22, -44, -52, -46, -56, -100, -44, -46, -96$.

Further, the maximum of $L(x)$ for $1{,}500 \leqslant x \leqslant 10{,}000$ occurs at $x = 3281$ when $L(x) = -7$, while $L(x)$ is, in the same range, minimum for $x = 9840$ when $L(x) = -130$.

S RAJARAMAN.

[Received 24-2-34].

Notes on the Theory of numbers (II). Remarks on the preceding Paper.

The following are some of the outstanding 'asymptotic' properties of $L(x)$.

(I) $$L(x) \leqslant 0 \qquad (x \geqslant 2)$$

implies† *the Riemann hypothesis on the roots of the zeta-function.*

(II) $$L(x) - \frac{\sqrt{x}}{\zeta(\frac{1}{2})} = \Omega_{R \atop L}(x^{\frac{1}{2}-\delta}), \quad \textit{for every } \delta > 0$$

[Note that $\zeta(\frac{1}{2}) < 0$].

(III) $$L(x) \sim \sqrt{x} / \zeta(\tfrac{1}{2}) \quad \textit{is false}‡.$$

(I) and (II) follow from Landau, *Handbuch*, Satz (S. 698).

(III) follows from Landau's *Vorlesungen*, II, S. 162.

S. CHOWLA.

[Received 24-2-34].

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* At the suggestion of Dr. S. Chowla.

† This was pointed out by Dr. A. Weil, who drew my attention to the problem (1) is mentioned by Polya, *loc. cit.*

‡ A proof of (III) was indicated to me by Dr. Walfisz.

On Euler's discussion of the equation $A^4 + B^4 + C^4 + D^4 = E^4$.

In this note, I show that Euler's discussion* of the problem

to find four biquadrates whose sum is a biquadrate

is apparently pointless.

Evidently $\quad A^4 + B^4 + C^4 + D^4 = E^4$

if $\quad A^2 = (p^2 + q^2 + r^2 - s^2)/n, \; B^2 = 2ps/n,$

$C^2 = 2qs/n, \; D^2 = 2rs/n, \; E^2 = (p^2 + q^2 + r^2 + s^2)/n.$

These five functions are to be made squares. This will be true of the first and last if

$$(p^2 + q^2 + r^2)/n = a^2 + b^2, \quad s^2/n = 2ab \qquad \ldots \quad (1)$$

Then $s^2 = 2abn$ = a square, if $2n = lm$, $a = lf^2$, $b = mg^2$, whence $a = lmfg$. Next,

$$2ps/n = 4pfg = 4x^2, \quad 2qs/n = 4qfg = 4y^2, \quad 2rs/n = 4rfg = 4z^2,$$

if $\qquad p = x^2/fg, \quad q = y^2/fg, \quad r = z^2/fg.$

Substituting these values into (1) we get

$$x^4 + y^4 + z^4 = \tfrac{1}{2}ab\,(a^2 + b^2)\dagger \quad \ldots \qquad \ldots \quad (2)$$

This is the final condition arrived at by Euler, and no discussion of this condition is given.

The right hand side of (2) is exactly equal to $\left(\frac{a+b}{2}\right)^4 - \left(\frac{a-b}{2}\right)^4$ and therefore the condition (2) can be written

$$x^4 + y^4 + z^4 + \left(\frac{a-b}{2}\right)^4 = \left(\frac{a+b}{2}\right)^4$$

which is of the same type as the original form, and hence Euler's discussion leads nowhere.

Vizagapatam K. SUBBA RAO.

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* Dickson's *History of the Theory of Numbers*, Vol. II, p. 649.

† The above discusssion is from Dickson, *loc. cit.*

On some Loci connected with a Triangle.

1. Given the base BC and the vertical angle A of a triangle, the following loci are discussed in text-books :

(1) *The Ortho-centre.* It is frequently stated that the locus is the arc of a segment of a circle on BC containing an angle equal to the supplement of A. The statement is erroneous, being incomplete when A is acute, and only partially true when A is obtuse. The actual locus is the arc of a segment of a circle containing an angle equal to A, described on a line equal and parallel to the given base and at a determinable distance from it.

(2) *The Ex-centre* I_1. The angle $BI_1C = 90° - A/2$ and therefore I_1 is on the segment of a circle on BC subtending an angle $90° - A/2$. The whole arc of this segment, however, does not form the locus. As I_1 lies on the external bisectors of the angles ABC and ACB, BI_1 and CI_1 cannot make with BC angles greater than a right angle. Hence I_1 cannot lie outside the region formed by the perpendiculars through B and C to BC. This restriction does not appear to have been noted before.

(3) *The Ex-centres* I_2 *and* I_3. I_2 and I_3 lie on the segment of a circle on BC containing an angle $A/2$. In this case, the segment within the region of the perpendiculars through B and C to BC cannot form part of the locus. The locus consists of two equal (detached) segments of a circle, one of which represents the locus of I_2 and the other of I_3. For a fuller discussion of this locus reference may be made to *The Mathematical Gazette*, Vol. VIII, Notes, pp. 263—265.

Of course it may be said from considerations of symmetry, that the locus includes a symmetrical counterpart on the other side of BC in all the above cases.

2. THEOREM. The locus of a point which is such that the rectangle under its differences from the equal sides of an isosceles triangle is equal to the square on its distance from the base is the circle touching the equal sides at the extremities of the base.

The theorem will be true only if the equality is true both in magnitude and in sign. If, however, the magnitude alone is taken into consideration, the proposition and the usual demonstration are

incomplete, for the ex-centres of the triangle corresponding to the equal sides satisfy the given condition and must be on the locus. In this case the locus of the point using the trilinears is given by

$$(\alpha^2 - \beta\gamma)(\alpha^2 + \beta\gamma) = 0$$

giving the circle and a hyperbola, having double contact with it at the ends of the base.

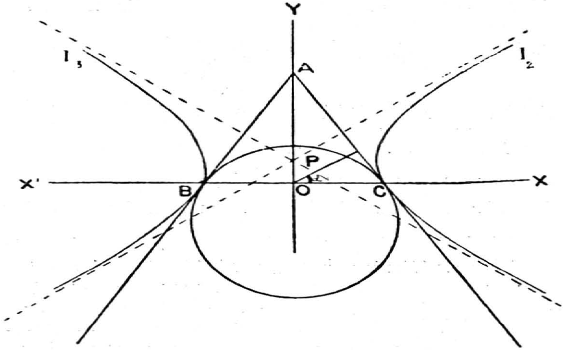
With the base and the bisector of the vertical angle of the triangle as co-ordinate axes, the equations to the sides may be written in the form $\pm x \cos\alpha + y \sin\alpha = p$. The equation to the required locus is then

$$y^2 = \pm (x\cos\alpha + y\sin\alpha - p)(y\sin\alpha - x\cos\alpha - p),$$

i.e., $x^2 + y^2 + 2py\tan\alpha\sec\alpha - p^2\sec^2\alpha = 0$ a circle

and $x^2\cos^2\alpha - y^2(1 + \sin^2\alpha) + 2py\sin\alpha - p^2 = 0$ a hyperbola

symmetrical about the y-axis. That the hyperbola passes through the ex-centres corresponding to the equal sides can be easily verified. The position of the centre and asymptotes and the magnitude of the axes and the eccentricity can be determined; the hyperbola is as drawn in the figure.



It may be remarked that as α varies from 90° to 0° the shape of the hyperbola changes, the eccentricity $\sqrt{2/(1 + \sin^2\alpha)}$ increasing from 1 to $\sqrt{2}$. In fact, given any hyperbola whose eccentricity is less

than $\sqrt{2}$, *i.e.*, whose transverse axis is greater than the conjugate axis, it is possible to find an isosceles triangle for which the hyperbola will be part of the locus considered.

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1, \qquad (a > b),$$

is the hyperbola and $y = \pm mx + c$ and $y = k$ the sides of the triangle. Comparing the equation of the hyperbola with

$$(y - mx - c)(y + mx - c) = -(1 + m^2)(y - k)^2$$

the values of m, k and c are

$$\pm \frac{b\sqrt{2}}{\sqrt{a^2 - b^2}}, \quad \pm b\sqrt{\frac{a^2 - b^2}{a^2 + b^2}} \quad \text{and} \quad \pm b\sqrt{\frac{a^2 + b^2}{a^2 - b^2}}.$$

It can be easily verified that the sides are tangents to the hyperbola at the points where the base meets it. Though these values apparently give two triangles, there is really only one triangle as they are symmetrically placed with respect to the x-axis.

If $a > b$, such a triangle is impossible. Hence all hyperbolas whose eccentricities lie between 1 and $\sqrt{2}$ may be considered as loci of points satisfying the condition $-p^2_a = p_b \cdot p_c$ for a properly chosen isosceles triangle. This result, I believe, is new.

Annamalai University,
Annamalainagar.

G. V. KRISHNASWAMI.

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22. If we compare a mathematical problem with an immense rock, whose interior we wish to penetrate, then the work of the Greek mathematicians appears to us like that of a robust stone-cutter who with indefatigable perseverance attempts to demolish the rock gradually from the outside by means of hammer and chisel; but the modern mathematician resembles an expert miner, who first constructs a few passages through the rock and then explodes it with a single blast, bringing to light its inner treasures.

HANKEL: Die Entwickelung der Mathematik in den letzen Jahrhunderten, (from *Memorabilia Mathematica*, (718).

On a property relating to an Ortho-centric tetrad.

1. It is known* that :

If from any point on a conic parallels be drawn to the diameters bisecting the sides of any inscribed triangle, the lines so drawn meet the corresponding sides of the triangle in three collinear points. ... (1.1)

In particular, when the four points form an ortho-centric tetrad on the conic, the conic is necessarily a rectangular hyperbola whose centre lies on the nine-point circle; and we have the result contained in question 1576,† *viz.*,

If O be any point on the nine-point circle of a triangle ABC whose ortho-centre is D and L, M, N the mid. points of BC, CA, AB, the parallels through D to OL, OM, ON meet the sides of ABC in three collinear points. ... (1.2)

It follows from (1.2) that if DX, DY, DZ is one triad of lines through D which meet the sides of ABC in three collinear points, then every triad of lines through D obtained by rotating the lines of the first triad through the same angle has this property.

Since the nine-point circle of ABC is symmetrically related to the ortho-centric tetrad ABCD, the property (1.2) relating to the triangle ABC is true also for the three other triangles formed out of ABCD. Thus we have four lines of collinearity L_1, L_2, L_3, L_4 by taking the triangles BCD, etc., respectively.

2. Let H be a unique rectangular hyperbola through ABCD having O for centre (the whole figure lying in a plane P). By projecting the points at infinity on H into the circular points at infinity on to the plane of projection p and denoting the projections by corresponding small letters, it is easily seen that a, b, c, d are points on a circle h in the plane p.

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* Taylor : *Geometry of Conics*, p. 366.

† Vide *The Mathematics Student*, Vol. I, p. 76.

The lines L_1, L_2, L_3, L_4 project into the pedal lines l_1, l_2, l_3, l_4 of a, b, c, d respectively with respect to bcd, etc. Since these four pedal lines meet at a point t, which bisects the join of any one of the points a, b, c, d to the ortho-centres of the triangles formed by the other three, we have the corresponding result in plane P, *viz.* :

If ABCD is any ortho-centric tetrad whose common nine-point circle is Γ, the lines L_1, L_2, L_3, L_4 are concurrent at a point T. ... (2.1)

Taking the triangle ABC, we note that the circum-circle Ω of ABC in P projects into the unique rectangular hyperbola ω in p through a, b, c, i, j where i, j are the projections on p of the circular points at infinity in P. This rectangular hyperbola ω necessarily passes through the ortho-centre u of abc. Hence, it follows that in the plane P, the parallels through A, B, C to OL, OM, ON meet at a point U on Ω. Since t is the mid. point of du in p, T is the mid. point of DU in P. Since the join of any point on the circum-circle of a triangle to its ortho-centre is bisected by its nine-point circle, we see that T is a point on Γ. Also from the similar triangles ABC, LMN (with centre of similitude G) it follows that if S is the centre of Ω, OS is parallel to DU and equal to half DU, so that O and T are collinear with the nine-point centre of ABC. Hence we arrive at the conclusion that

as O moves on Γ, the corresponding point T also moves on Γ, so that OT is a diameter of Γ. ... (2.2)

If we take the point T as our starting point, the above process leads to O, as the point of concurrence, so that the relation between O and T is symmetric.

Annamalainagar. K. RANGASWAMI.

REVIEWS.

The Calculus of Finite Differences, by L M. MILNE-THOMSON, M.A., F.R.S.E., Assistant Professor of Mathematics in the Royal Naval College, Woolwich. pp. xxiii + 558. London. 1933. (Macmillan & Co.) *Price* 14*s*.

The necessity for a good text-book and an introduction to the neglected subject of Finite Differences has been felt in this country as in all others where English is the medium of mathematical instruction. With us, the long-vanished treatise of Boole still figures in the average syllabus, and the fact of its being out of print is hardly felt, as no one ever gives the calculus the attention that it deserves.

The author has kept both the above needs in mind, and so far as a text-book to supplant Boole is concerned, has succeeded admirably. The reason for this is seen in the first seven chapters, of which five are devoted to interpolation. Prof. Milne-Thomson's tables of elliptic functions fully prepare us for the excellence of this part. Chapters VI and VII treat Bernoullian and Eulerian polynomials, and finite differentiation; the writer himself considers these seven chapters a fair introduction to the subject. Had the work ended here, the student would have had a moderately priced hand-book, particularly useful to the computer. It is gratifying, for example, to see the Gaussian interpolation formula properly treated, whereas, in the usual order of things, it is fired off at the class as a property of Legendre polynomials.

Unfortunately, the rest of the work is "room ... for the more modern developments of the finite calculus." The treatise of Norlund might be too heavy for the non-polyglot beginner whom the author always tries to keep in mind. But it does not seem worth while to write for such a person, if the result is to be so sad a limitation of the actual theory presented. To treat linear difference equations without mentioning Birkhoff, or the gamma function without so much as a reference to Mellin—let alone his inversion formulae—these seem to us to diminish the importance of the already neglected difference calculus. The inclusion or exclusion of any particular topic must always rest with the author, who, undoubtedly has his reasons for the use of the

"Auswahlprinzip" in this sense. But the full reach of the modern developments of the subject in question might perhaps have been shown with more relief by some mention of the direct method of the calculus of variations; of the work of Schürer in mixed difference and differential equations. Perhaps a treatment of the Heaviside-Carson operational calculus could have been given by means of mixed equations in two variables, and might have added to the value of the work for physicists as well as engineers. Not only does one regret the omission of such really stimulating questions, but one feels at times that the stress laid on pure application is excessive. Even in interpolation formulae—the writer's strong point—one does not gain insight into the actual mechanism of the processes involved; whatever the value of theory may be for computation, some of us at least will always prefer such expositions as, for instance, are typified by the lecture of Fejér published in the *American Mathematical Monthly* (XLI, 1934, pp. 1-14).

It is to be hoped that the author sees fit, in some later edition to issue the first two hundred pages separately as an elementary text; and to do real justice to the "modern developments" as well as to his own undoubted talents in a second volume.

D. D. K.

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23. At a considerably higher level we find the contemporary mathematician who has still to learn the real meaning of "experimental verification," and who is habituated to treating the schemes of concepts in his brain as truer than fact, at odds with the modern biologist. Still constrained in the logical net, he shakes his head at the "unphilosophical" ease of the latter's mental movements. He objects to conclusions that are not final and exactly proved. He has not learned to rest in a provisional conolusion, and clings to the delusion that purely symbolical processes can win truth from the unknown. His symbolic processes never do win truth from the unknown, but he fancies that they justify an attitude of disapproval towards the pragmatical acceptances of practical science. But in the long run even the mathematicians will become scientific.

H. G. WELLS: *The Work, Wealth and Happiness of Mankind*, p. 69

SOLUTIONS TO QUESTIONS.

Question 1605.

(K. J. SANJANA AND K F. VAKIL):—In connection with Mr. R. Ramanujan's Question No. 1598 (*J. I. M. S.*, August 1931) prove the following two results.

(1) The length of the side BC of the pentagon ABCDE required to be constructed is given by the equation

$$BC \, . \, DG = 16992 \times (3 \, . \, 5 \, . \, 7 \, . \, 13 \, . \, 47 \, . \, 59 \, . \, 61 \, . \, 83 \, . \, 317)^{\frac{1}{2}}.$$

(2) If P and Q are the lengths of the perpendiculars drawn from the point G respectively upon CD and BF, then $P/Q = 2867/437$.

Solution by V. A. Mahalingam, M. S Srinivasachari and the proposer.

B, C, D, F are cyclic. Hence by Ptolemy's Theorem,

$$BF.CD + BC.DF = BD.CF.$$

$$\therefore \quad BC.DF = BD.CF - BF.CD = CD(CF - BF)$$

since $BD = CD$.

Also, since the triangles BGC and FGD are congruent,

$$\frac{DF}{BC} = \frac{GF}{BG} = \frac{83}{59} \quad \textit{i.e.,} \quad DF = \frac{83}{59} \, . \, BC.$$

Substituting this for BF, and using the value of CD we have, after simplification,

$$BC^2 = \frac{59 \, . \, 2^{10} \, . \, 13 \, . \, 317 \, , \, 5 \, . \, 3^5 \, . \, 59^2 \, . \, 7}{83 \, . \, 2867}$$

Also it is given that $DG^2 = 83^2 \, . \, 2867^2$

$$\therefore \quad BC \, . \, DG = 16992 \, . \, (3 \, . \, 5 \, . \, 7 \, . \, 13 \, . \, 47 \, . \, 59 \, . \, 61 \, . \, 83 \, . \, 317)^{\frac{1}{2}}.$$

Lastly, the triangles GCD and GBF are similar and their altitudes are, therefore, as the corresponding sides. Hence

$$\frac{P}{Q} = \frac{CD}{BF} = \frac{GC}{BG} = \frac{59 \, . \, 2867}{19 \, . \, 23 \, . \, 59} = \frac{2867}{437}.$$

Question 1615.

(P. JAGANNATHAN) :—Prove that

$$\frac{2 \cdot (3n)!}{n!\,(n+1)!\,(n+2)!}$$

is an integer for all positive integral values of n.

Examine whether the factor 2 in the numerator may be omitted without affecting the result.

Solution by Hansraj Gupta.

Let p be any prime number. Then the number of times that p occurs as a factor of $q!$ is given by the expression

$$\sum_{k=1}^{\infty} [\, q/p^k \,] = [\, q/p \,] + [\, q/p^2 \,] + [\, q/p^3 \,] + \dots \dots,$$

where $[x]$ denotes the integral part of x. Hence p occurs as a factor of the denominator

$$\sum_{k=1}^{\infty} \{ [\, n/p^k \,] + [\, (n+1)/p^k \,] + [\, (n+2)/p^k \,] \}$$

times. It occurs as a factor of the numerator $\sum_{k=1}^{\infty} [\, 3n/p^k]$ times.

Now

$$[\, n/m \,] + [\, (n+1)/m \,] + [\, (n+2)/m \,] \leqslant [\, 3n/m \,]$$

for all values of $m \geqslant 3$; for let $n = am + n_1$, where $n_1 < m$. Then $[\, n/m \,] + [\, (n+1)/m \,] + [\, (n+2)/m \,] = 3a,\ 3a+1$, or $3a+2$ according as $n_1 <, =$ or $> m-2$; and

$$\left[\frac{3n}{m}\right] = \left[\frac{3am + 3n_1}{m}\right] \geqslant 3a, \qquad \text{if } n_1 < m-2,$$

$$= \left[\frac{3am + m + (2m-6)}{m}\right] \geqslant 3a+1, \text{ if } n_1 = m-2,\ m \geqslant 2,$$

$$= \left[\frac{3am + 2m + (m-3)}{m}\right] = 3a+2, \text{ if } n_1 = m-1,\ m \geqslant 3.$$

Also

$$[\, n/2 \,] + [\, (n+1)/2] + [\, (n+2)/2 \,] = 1 + [\, 3n/2 \,]$$

for, if $n = 2a$, then

$$[n/2] + [(n+1)]/2] + [(n+2)/2] = 3a+1 = 1 + [3n/2] \dots \text{(A)}$$

while if $n = 2a+1$, it is $3a+2 = 1 + [3n/2]$.

These lemmas go to prove the given result.

From (A) it is clear that the factor 2 is necessary in the numerator. If, however, $n \geqslant 3$, then for some value of k, $2^{k-1} \leqslant n+2 < 2^k$. Hence, $[n/2k] + [(n+1)/2k] + [(n+2)/2^k] = 0$, while $[3n/2k] = 1$. So in this case the factor 2 is not necessary in the numerator.

Also solved by S. Minakshisundaram and the proposer.

Question 1639.

(K. SATYANARAYANA) :—Show that the locus of the centre of a conic with axes in given directions, and for which a given triangle is self-conjugate, is a rectangular hyperbola passing through the vertices whose asymptotes are parallel to the axes of the conics.

Solution by Altshiller Court, Hukam Chand, S. Pichai, V. Ramaswamy Aiyar, R. Ranganatha Rao, K. Rangaswami, H. S. Subba Rao and G. Sankara Iyer.

If O be the centre of such a conic and ABC the given triangle and P, Q the points at infinity in the given directions, then ABC, OPQ are both self-polar with regard to the same conic. Hence all the six points lie on a conic, *i.e.*, O lies on the conic determined by ABCPQ which is the rectangular hyperbola described in the question.

Analytical Solution by Hansraj Gupta, Hariharan, Hukam Chand, S. P. Ranganathachar, V. Ranganathan and Vidyachandra.

Question 1642.

(C. N. SRINIVASAIENGAR) : —At any point of a non-torsal generator of a scroll, the scroll and the limiting conicoid through the generator and two generators consecutive to it have the same principal radii of curvature. Also if any two surfaces touch along a generator, their Gaussian curvatures at any point along it are equal.

Solutions (1) by the Proposer; (2) by K. Rangaswami.

(1) For any surface, the two asymptotic lines through any point have the same torsion numerically, *viz.*, $\sqrt{K}$ where K is the Gaussian curvature. The proof of this result, called Enneper's formula, depends upon the property that the binormals to the curve at any poin coincides with the normal to the surface thereat. Hence, in the case of a generator of a surface (not necessarily a scroll), if we define the binormal as the normal of the surface, the torsion of the generator at any point will be given by the same formula. (There will be no difficulty in seeing that the Serret-Frenet formulae continue to hold for the straight line with this definition of the binormal). The second part of the question now follows.

For a given generator of a scroll, there are ∞^3 conicoids which touch the scroll along the generator. For the particular conicoid considered, both the inflexional tangents at any point on the generator are the same as for the scroll. Now, the semi-angle between the inflexional tangents is given by $\tan^{-1}\sqrt{-\rho_2/\rho_1}$. Therefore $\rho_1\rho_2$ as well as ρ_2/ρ_1 are the same for both the surfaces. Hence the result.

(2) The first part is proved in Bell's *Co-ordinate Geometry*, p. 346.

At any point of a generator g distant x from the central point, the value of the Gaussian curvature is $-p^2/(p^2 + x^2)^2$ where p is the parameter distribution for the generator (*vide* Bell: *Co. Geom.*, p. 348). Also if two surfaces touch along a generator, the central point and the parameter of distribution of the generator are the same for both the surfaces. Hence their Gaussian curvatures along g are equal.

Also solved by P. Jagannathan, V. T. Srinivasan.

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* BELL: *Co-ordinate Geometry*, § 245; Weatherburn: *Differential Geometry* 38. Also, *vide*, J. I. M. S., Vol. III, p. 86.

ANNOUNCEMENTS AND NEWS.

Readers are invited to contribute to the value of this section by sending suitable news items of interest.

In accordance with our declared policy of publishing from time to time portraits of Mathematicians who have rendered conspicuous service to the student world or have made noteworthy contributions to the subject, we have great pleasure in including in this issue a portrait of Prof. G. N. Watson, Sc.D., F.R.S. There are few students of higher mathematics in Indian Universities to whom Prof. Watson's classical joint work on ***Modern Analysis*** has not been a source of help and inspiration; while his encyclopaedic treatise on ***Bessel Functions*** is a mine of information to serious students of the subject.

The Managing Committee of the Society has passed a resolution recording its appreication of the valuable services rendered by Mr. S. R. Ranganathan as Hon. Treasurer of the Society for nearly six years.

The Committee has approved of the following concessions regarding Life-composition fees where the membership has been continuous and long-standing; namely, Rs. 100, Rs. 75, Rs. 50 and Rs. 25 respectively for those whose membership extends continuously over 10, 15, 20 and 25 years. Members of 30 years' standing are exempt from all fees. The normal life composition fee is Rs. 150.

These concessions are made in deference to the wishes of the Members of the Society as expressed at the general body meeting held at the Jubilee Conference.

The Society acknowledges with gratitude the receipt of financial help from the Madras, Bombay and Annamalai Universities to the extent mentioned below :

		Year of receipt.
Madras University	Rs. 350	1933
Bombay University	,, 200	1934
Annamalai University	,, 100	1934

Dr. Alfred Moessner in Nurnberg, Germany, Nordring 38, wishes to announce that he would gladly enter into correspondence with all readers of the STUDENT interested in the Theory of Numbers.

In recognition of his long-continued and important contributions to the cause of American Mathematics the members of the Mathematical Association of America have voted unanimously to suspend the bye-laws and to elect Professor H. E. Slaught Honorary President of the Association for life.

Sir James Jeans has been elected President of the British Association for the year. The Presidential address is to be delivered at Aberdeen in September.

The tenth International Congress of Actuaries will be held at Rome between the 4th and the 10th of May, 1934.

Sir C. V. Raman has been invited to deliver the inaugural address at the first International Congress of Radiobiology at Venice in September 1934.

The Cambridge University has awarded its Smiths prizes to K. Mitchell, Esq., B.A., of Peterhouse and T. Ward, Esq., of Emmanuel College, and its Raleigh Prizes to M. S. Bartlett, Esq., B.A., of Queen's College and C. G. Pendse, Esq., B.A., of Downing College. Mr. C. G. Pendse is the son of Prof. G. G. Pendse of Baroda College, a member of the Society, and his essay was on the Theory of Saturn's Rings. We offer our felicitations to the prize-winners.

Prof. G D. Birkhoff of the Harvard University has been awarded a prize of 1000 lire donated by Pope Pius XI for the best book on "system for the solution of differential equations."

At the next International Conference of Mathematicians to be held in Oslo (formerly Chiristiania), Norway in 1936, two gold medals are to be awarded to out-standing mathematicians. The foundation is due to Dr. J. C. Fields, President of the Toronto Mathematical Congress (1924) and is financed out of the proceeds remaining from the sale of the *Proceedings* of the Toronto Congress.

The Jeypore Vikram Deo College of Science and Technology and the University College of Arts were opened by the Chancellor, Andhra University, in December 1933.

Mr. Gian Chand, M.A., has been appointed Lecturer in Mathematics, Ramja's College (Delhi University).

K. Nagabhushanam, Esq, research scholar in the Andhra University, has been appointed Lecturer in Mathematics in the same University.

Sir Thomas Muir died on 21st March 1934. By a deed of gift he has bequeathed his library to the South African state

The Indian Mathematical Society

Statement of Accounts for the Year 1933,

RECEIPTS.

				Rs.	A.	P.
Balance from 1932						
Savings Bank a/c ...	2,916	8	11			
Current Account ...	147	2	6			
Fixed Deposit ...	3,800	0	0			
				6,863	11	5
Subscription from Members ...				1,189	0	0
Life Subscription (from Prof. Panday) ...				75	0	0
Subscription for Journal ...				157	3	0
Grant in Aid (from the Madras University)				350	0	0
Interest on investments ...				196	6	5
Silver Jubilee Conference ...				500	0	0
Miscellaneous ...				32	12	6
Total ...				9,364	1	4

EXPENDITURE.

				Rs.	A.	P.
Ordinary Working Expenses ...				210	7	3
Books and Periodicals ...				765	14	0
Journal and other Printing ...				655	3	0
Library Expenses ...				450	0	0
Closing Balance :—						
Fixed Deposit ...	5,600	0	0			
Savings Bank a/c ...	1,540	2	10			
Current Account ...	142	6	3			
				7,282	9	1
Total ...				9,364	1	4

(Sd.) L. N. SUBRAMANIAM,
Hony. Treasurer.

Audited and found correct.

(Sd.) G. A. SRINIVASAN,
Hony. Auditor.

QUESTIONS FOR SOLUTION.

Proposers of Questions are requested to send their own Solutions along with their Questions.

1668. (ALFRED MOESSNER, Nurnberg, Germany) :—

(i) We have

$$17^n + 17^n = (-3)^n + 5^n + 12^n + 20^n \qquad (n = 1, 2, 3).$$

Is the equation $A_1^n + A_2^{2n} = B_1^n + B_2^n + B_3^n \quad (n = 1, 2, 3)$ also solvable when $A_1 \neq A_2$? What is the general solution?

(ii) We have $7^n + 7^n = 3^n + 5^n + 8^n \qquad (n = 2, 4)$.

Is the equation $E_1^n + E_2^n = F_1^n + F_2^n + F_3^n \quad (n = 2, 4)$ also solvable for $E_1 \neq E_2$?

1669. (HANSRAJ GUPTA) :—If $|x| < 1$, prove that

$$1/(1-x^2)(1-x^3)(1-x^4)\ldots = 1 + \sum_{s=1}^{\infty} x^{2s}/(1-x)(1-x^s)\ldots(1-x^s).$$

1670. (B. RAMAMURTI) :—If $f_r(x)$, $r = 1, 2, 3, 4$, be four linearly independent polynomials of the same degree, and J_{rs} denotes the Jacobian of $f_r(x)$ and $f_s(x)$ prove that

$$\Sigma\,[J_{12}(x)\,J_{34}(y) + J_{12}(y)\,J_{34}(x)]$$

has the factor $(x - y)^4$, the summation corresponding to the permutations (12) (34), (23) (14) and (31) (24).

Hence, or otherwise, prove that if r_1, r_2, r_3, r_4 be four distinct integers, the equation $\Sigma\,(x^{r_1+r_2-1} + x^{r_3+r_4-1})(r_1-r_2)(r_3-r_4)=0$ has unity as a quadruple root.

1671. (NATHAN ALTSHILLER COURT, Oklahama, U.S.A.) :— Consider the variable line on which two fixed spheres determine two harmonic segments. The locus of the mid. points of these segments is a sphere.

1672. (V. RAMASWAMI AIYAR) :—ABCD is a quadrilateral inscribed in a cirle, centre O. BC, AD intersect at P. If X, Y are the circum-centres of the triangles ABP, CDP, show that the circum-radii of these triangles are equal to OY, OX.

1673. (V. RAMASWAMI AIYAR) :—A triangle $\alpha\beta\gamma$ moves rigidly and without rotation, so as to be constantly in perspective with a given triangle $A_0B_0C_0$. Prove that there is an infinity of fixed triangles ABC with respect to each of which the moving triangle $\alpha\beta\gamma$ is constantly in perspective. Prove further that the locus of the centre of perspective, corresponding to different triangles ABC, forms a system of concentric, similar and similarly situated conics [briefly *co-asymptotic conics*].

1674. (T. R. RAGHAVA SASTRY) :—O is a fixed point in the plane of a triangle ABC. The reciprocal of ABC with respect to O is the triangle PQR. It is stated in Question 1631 that the centre of perspective of the triangles ABC and PQR for different radii of reciprocation traces out the rectangular hyperbola through ABCO.

Show further that the axis of perspective of the two triangles envelops a parabola inscribed in ABC and having the line joining O to the ortho-centre of ABC as directrix.

1675. (A. RANGANATHA RAO) :—A point P and a line l are pole and polar with respect to each of two conics S, S′. If *one* chord AB of S′, which is a tangent to S be such that PA, PB are conjugate lines with respect to S, then the same is true of *all* chords of S′ which are tangents to S.

1676. (A. A. KRISHNASWAMY AYYANGAR) :—Show that there is a unique solution to the letters A, E, G, S and the missing digits, which will make the following multiplication true. No two different letters represent the same digit.

			E	*	*	*	S
					A	G	E
		*	*	*	E	G	*
		*	*	G	*	E	
G	E	*	*	*	A		
G	A	*	*	E	*	*	*

RECENT PUBLICATIONS.

We are opening this column to bring important new publications in Mathematics to the notice of our readers from time to time.*

BAKER Principles of Geometry, Vols. 5 and 6; Cambridge University Press.

BERNSTEIN Lecons sur les progres recents de la theorie des series de Dirichlet (Preface by HADAMARD), Collection Borel, Gauthier Villars, Paris.

BIRKHOFF Aesthetic Measure, pp. 226, Harvard University Press. $7.5.

CARTAN Sur la structure des groupes de transformations finis et, continus; Vuibert, Paris.

EECKE Pappus d'Alexandrie, Tr from the Greek, De Brouwer, Paris and Bruges.

EINSTEIN Les fundaments de la theorie de la relativite generale; Theorie unitaire de la gravite et de l'electricite; Sur la structure cosmologique, Tr. by Solovine, Herman, Paris.

EULER Opera Omnia, Vol. 16, Teubner.

HARDY, LITTLEWOOD and POLYA Inequalities. Cambridge, 18*s*.

HILBERT and BERNAYS Grundlagen der Mathematik (Grundlagen d. Math. Wiss. No. 40) Springer.

HURWITZ Werke, Basel Buchhauser.

INGE God and the Astronomers, Longmans.

MACPIKE Halley, correspondence and papers, Oxford, 21*s*.

MERRIL Mathematical excursions, The Norwood Press.

MITCHELL Elements of mathematical Analysis, 600 pp. Macmillan.

MORLEY Inversive Geometry, G. Bell and Sons, 16*s*.

PHILLIPS Vector Analysis, John Wiley and Sons, N. Y.

SOMERVILE Analytical Geometry of three dimensions, Cambridge. 18*s*.

THOMAS Differential invariants of generalised spaces, Camb. 21*s*.

WALKER Conjugate functions for beginners, Oxford.

WHITEHEAD Adventures of ideas, Cambridge, 12*s*. 6*d*.

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* Publishers of mathematical books are requested to give the Editor details regarding their publications with a view to insertion in this column, if space permits. Details regarding the contents of the books can only be accepted as advertisements.

List of periodicals received in the Library of the Indian Mathematical Society, during the months of February & March 1934.

Acta Mathematica. **62**, *1, 2.*

HAMBURGER, Ribaucourtransformationen und spharische Abbildung MAHLER, KURT, Zur Approximation algebraischer Zahlen III (Uber die mittlere Anzahl der Darstellungen grosser Zahlen durch binare Formen) TRJITZINSKY, W. J., Analytic theory of linear differential equations.

American Journal of Mathematics. **56**, *1. (3 Copies).*

American Mathematical Monthly. **41**, *1, 2. (3 Copies each).*

On the characterization of some remarkable systems of points of interpolation by means of conjugate points; By L. FEJER; Thermodynamics —an Exposition, II; By J. E. TREVOR. Evaluations of certain Definite Integrals, by G. RUTLEDGE and R. D. DOUGLAS; Simplification of the equations of Conics, by H. B. THORNTON.

The Convergence of Fourier Series: By JACKSON; The Postulational Method in Mathematics; By E. V. HUNTINGTON.

Astrophysical Journal, **79**, *1.*

Boletin del Seminario Matematico. **3**, *13.*

Bulletin of the American Mathematical Society. **40**, *2.*

Note on a Problem of Frechet, By R. STEPHENS. On the Summation of Fourier Series, by R. E. A. C. PALEY, W. C RANDELS, and M F. ROSSKOPF. Curvatures in Riemannian Space, By HARRY LEVY. A Note on a certain property of a Family of Curves, by A. WERTHEIMER. Note on Homogeneous Functionals. By L. S. KENNISON. Note on the Orthogonality of Tchebycheff Polynomials on Confocal Ellipses, By J. L WALSH. Derivatives, Difference Quotients and Taylor's Formula by WHITNEY. Linear Associative Algebra of Infinite order whose elements satisfy finite algebraic equations, by H. H. CONWELL. The apparent contour of the general V_3 in S_4 by A. R. WILLIAMS. On power characters in Cyclotomic Fields, by H. S. VANDIVER. Fermat's last theorem and the second factor in the cyclotomic Class Number, by H. S. VANDIVER. Independent Postulates for the "Informal" part of Principia Mathematica, by E. V. HUNTINGTON. Independent Postulates for an "Informal Principia System with Equality" by E. V. HUNTINGTON. A cyclic involution of order seven, by W. R. HUTCHERSON. Note on relations connecting certain cases of convergence in the Mean, by D. JACKSON. Concerning maximal sets, by G. T. WHYBURN. Integral domains of rational Generalized Quaternion Algebras. By A. A. ALBERT.

Bulletin des Sciences Mathematiques. 58, Jan. 1934.

Crelle's Journal. **170**, *3, 4.*

Current Science. **2**, *8, 9.*

Jahresbericht der deutschen Mathematiker Vereingung. **43**, *5-8.*

Mathematical Gazette. **18**, *227. (3 Copies).*

Presidential Address, 1934 : Scraps from some Mathematical Note-Book,. by G. N. WATSON. Pansymmetrical Pencils, by B. M. PEEK. Congruence and Parallelism, by E. H. NEVILLE. Correspondence. by J. M. CHILD. The Operators E and △, by S. BARNARD. On Recurring Continued Fractions, by H. ORFEUR. Theory of Exponential and Logarithmic Functions and their derivatives, by W. MILLER, Determination of the foci, directrices, axes and eccentricities of a conic whose Equation is given with numerical Co-efficients, by L. CRAWFORD.

Mathematische Annalen. **109**, *3.*

Monthly Notices of the Royal Astronomical Society. **94**, *2.*

Philosophical Magazaine. **17**, *110, 111, 112.*

Philosophical Transactions of the Royal Society of London. **232**, *717.*

Popular Astronomy. **42**, *1, 2.*

Proceedings of the Cambridge Philosophical Society. **30**, *1.*

Proceedings of the London Mathematical Society. **36**, *7. (2 Copies).*

On the representations of a number as a sum of squares and products (continued) : A. PAGE. On Epstein's Zeta-function : E. C. TITCHMARSH. Approximate equations for the Epstein Zeta function : H.S.A. POTTER. Theorems concerning Cesaro means of power series : G. H. HARDY and J. E. LITTLEWOOD. On the Moduli of regular functions : H FRAZER. A Boundary-value Problem for Harmonic function : W. G. L. SUTTON.

Proceedings of the Physico-Mathematical Society of Japan. **16**, *1, 2.*

Proceedings of the Royal Society of London. **143**, *850, 851.*

Quarterly Journal of Mathematics. **4**, *14, 15, 16.*

Revista Matematica Hispano Americana. **8**, *1, 2, 3, 4.*

Universidad Nacional de La Plata. No. 95.

BOOKS.

L. M. MILNE THOMSON, M.A., F.R.S.E.; The Calculus of Finite Differences.

190035

THE MATHEMATICAL ASSOCIATION OF AMERICA.

The Mathematical Association of America is devoted primarily to the interests of collegiate mathematics. It now has over thousand individual and institutional members. There are twenty sections at present, representing thirty-one different States. The Association holds two national meetings per year and the sections hold one or two meetings per year. All meetings, both national and sectional, are reported in the Official Journal.

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VOLUME II. NUMBER 1.

MARCH, 1934.

CONTENTS

	Pages
Vague Historical Views relating to the Negative Numbers : By **G. A. Miller**	1 — 6
On Powers of Numbers : By **K. Subba Rao**... ...	6 —10
A Biquadrate as the sum of four Biquadrates : By **S. Sastry**	10—11
A Brief Sketch of Hindu Trigonometry : By **D. A. Somayajulu**	12—21
Notes and Discussions :	
Congruence Properties of Partitions : By **S. Chowla** ...	22
On a certain Arithmetical Function : By **S. Rajaraman**	23
Notes on the Theory of numbers (II). Remarks on the preceding Paper : By **S. Chowla**	23
On Euler's discussion of the equation $A^4 + B^4 + C^4 + D^4 = E^4$: By **K. Subba Rao** ...	24
On some Loci connected with a Triangle : By **G. V. Krishnaswami**	25—27
On a property relating to an Ortho-centric tetrad : By **K. Rangaswami**	28—29
Gleanings	21
Reviews : **D. D. K.**	30—31
Solutions to Questions	32—35
Announcements and News	36—37
Questions for Solution	39—40

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